SPRING 2025 MATH 590: QUIZ 2

Name:

1. Let V be a vector space over F and $v_1, \ldots, v_r \in V$ be a finite set of vectors. State what it means for $v_1, \ldots, v_r$ to be *linearly dependent*, and then state an equivalent condition. (3 points)

Solution. Vectors $v_1, \ldots, v_r$ are linearly independent if some v_i belongs to $\mathrm{Span}\{v_1, \ldots \hat{v}_i, \ldots, v_r\}$. Equivalently, the vectors are linearly dependent if there is a non-trivial depndence relation $a_1 v_1 + \cdots + a_r v_r = \vec{0}$ with at least one $a_i \neq 0$. □

2. Determine if the vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix}$ are linearly dependent or linearly independent. If they are linearly dependent, provide a non-trivial dependence relation among them. (4 points) You may use the reverse side of this page if necessary.

Solution. The vectors are dependent, if the system of equation with augmented matrix has a non-trivial solution:

$$\left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 1 & 2 & 5 & 0 \\ 2 & -1 & 0 & 0 \end{array}\right] \xrightarrow{-1\cdot R_1+R_2, -2\cdot R_1+R_3} \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -7 & -14 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 7 & 14 & 0 \end{array}\right] \xrightarrow{3\cdots R_2+R_1, 7\cdot R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system has a non-trivial solution, so the given vectors are linearly dependent. There are infinitely many solutions to the system, and thus infinitely many non-trivial dependence relations. One such relation is $-1 \cdot v_1 - 2 \cdot v_2 + 1 \cdot v_2 = \vec{0}$. □

3. Let S_1, S_2 be linearly independent subsets of the vector space V. Must $S_1 \cup S_2$ be linearly independent? Justify your answer. (3 points)

Solution. $S_1 \cup S_2$ need not be linearly independent. There are many examples. For example, in $\mathbb{R}^2$, take $S_1 = \{(1,0), (0,1)\}$ and $S_2 = \{(1,1\}$. Then $S_1 \cup S_2 = \{(1,0), (0,1), (1,1)\}$ is not independent, since $(1,1) = (1,0) + (0,1)$. □